Plenty of Franklin Magic Squares, but none of order 12

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Abstract

We show that a genuine Franklin Magic Square of order 12 does not exist. This is done by choosing a representation of Franklin Magic Squares that allows for an exhaustive search of all order 12 candidate squares. We further use this new representation (in terms of polynomials) to generate large classes of true Franklin Magic Squares of orders 8 and multiples of 16. Next we show how Franklin Magic Squares of orders $n = 20 + 8k$ can be constructed. Finally we indicate how almost-Franklin Magic Squares of order 20 can be constructed in a general way.

1 Franklin Magic Squares

According to various descriptions a natural *Franklin Magic Square* of even size n is a square matrix M with n rows and columns with the properties

1. the entries of M are $1, 2, \ldots, n^2$;
2. each row and each column has a fixed entry sum $n(1 + n^2)/2$;
3. each two by two sub-square $\begin{bmatrix} M_{i,j} & M_{i,j+1} \\ M_{i+1,j} & M_{i+1,j+1} \end{bmatrix}$ has sum $2(1 + n^2)$;
4. each half row starting in column 1 or $n/2 + 1$ has sum of entries equal to $n(1 + n^2)/4$, and similar for half columns starting in row 1 or $n/2 + 1$;
5. each half of the main diagonal (starting in column 1 or $n/2 + 1$) together with each half of the back diagonal has total sum (such as $\sum_{i=1}^{n/2}(M_{i,i} + M_{i,n+1-i})$) equal to $n(1+n^2)/2$. This construction is called a *bent diagonal*. The sum requirements also hold for so-called *bent rows*, which are translates of the two half-diagonals, possibly wrapping over the matrix sides.

These squares are called after the former US president and scientist Benjamin Franklin who constructed a few of such matrices, two of order eight, one of order 16. Note that the fourth property implies that n is a multiple of four. It turns out that the condition on 2x2 subsquares is the most prominent one and generates a lot of shapes with a constant-sum-property. In particular we have that

2x2 squares wrapping along one side of the matrix also have fixed sum $2(1 + n^2)$,

and further that

for arbitrary i, j, k the entries in $\begin{bmatrix} M_{i,j} & M_{i,j+1+2k} \\ M_{i+1,j} & M_{i+1,j+1+2k} \end{bmatrix}$ have sum $2(1+n^2)$.

Combining this on two consecutive rows $i, i+1$ we find that for arbitrary i, j, k,

$M_{i,j} + M_{i,j+1+2k}$ and $M_{i+2,j} + M_{i+2,j+1+2k}$ have equal values.

In the end this leads to the observation that for arbitrary i, j, k, m we have that

each four-tuple $M_{i,j}, M_{i+2m+1,j}, M_{i,j+2k+1}, M_{i+2m+1,j+2k+1}$ has sum $2(1+n^2)$.

This property of Franklin Magic Squares is often referred to as the mirroring property because its consequence is that on any shape that is symmetric horizontally and vertically along a line separating rows or columns, respectively, the entries of square add up to a number that is independent of the choice of the intersection of the axes of symmetry. Here we allow moving over the border of the square by embedding it on a torus. Note that this property is merely based on the 2x2 sub-square property.

Applying the above insights on the top halves of the main and back diagonal we find that the sum-of-half-diagonals property is equivalent (given the 2x2 square with fixed sum) to the statement that $M_{1,1}, M_{2,2}, M_{1,3}, M_{2,4}, \ldots, M_{1,n/2-1}, M_{2,n/2}$ and $M_{2,n/2+1}, M_{1,n/2+2}, \ldots, M_{2,n-1}, M_{1,n}$ together sum up to $n(1+n^2)/2$.

Subtracting $n/4$ 'subsquares' $[M_{1,2k}, M_{2,2k}, M_{1,n+1-2k}, M_{2,n+1-2k}]$ of constant sum we find that $M_{1,1} - M_{1,2} + \ldots + M_{1,n/2-1} - M_{1,n/2}$ plus its mirror image $M_{1,n+1-1} - M_{1,n+1-2} + \ldots + M_{1,n+1-n/2-1} - M_{1,n+1-n/2}$ equals zero.

Adding a full row sum leads to a pattern of $n/2$ entries, with

$$(M_{1,1} + M_{1,3} + \ldots + M_{1,n/2-1}) + (M_{1,n/2+2} + M_{1,n/2+4} + \ldots + M_{1,n}) = n(1+n^2)/4.$$

which holds for ordinary magic squares with the 2x2 square property and the bent-diagonals-property.

Subtracting a *fixed half row sum* starting in column $n/2+1$ we finally obtain the property that

$M_{1,1} + M_{1,3} + \ldots + M_{1,n/2-1}$ equals $M_{1,n/2+1} + M_{1,n/2+3} + \ldots + M_{1,n-1}$.

This *alternate sum property* is hence equivalent with the bent-diagonal property (in presence of the other franklin conditions), but much more easily checked. Obviously a similar reasoning is possible for vertical bent-diagonals, leading to columns having the alternate sum property.

All this leads to the following more compact definition of a Franklin Magic Square of arbitrary order $4k$, which is a matrix with properties:

1. entries are $1, \ldots, n^2$;
2. each 2x2 sub-square has entries summing up to $2(1+n^2)$;
3. the first half of the first row, the second half of the first row, the first half of the first column, and the second half of the first column, each have entries that sum up to $n(1+n^2)/4$;

4. entries on odd positions in the first half of the first row add up to the same value as entries on odd positions in the second half of the first row; similarly, entries on odd positions in the first half of the first column add up to the same value as entries on odd positions in the second half of the first column.

2 More compact representation

2.1 Isomorphisms

It turns out that any Franklin Magic Square maintains its magic properties under a number of matrix transformations, namely:

1. reflection along the horizontal, or vertical axis of symmetry;
2. permutation of row (column) indices within the sets $S_1 = \{2k+1 \,|\, 0 \le k < n/4\}$, $S_2 = \{2k \,|\, 1 \le k \le n/4\}$, $S_3 = \{2k+1 \,|\, n/4 \le k < n/2\}$ and $S_4 = \{2k \,|\, n/4 < k \le n/2\}$;
3. exchanging the $n/4$ rows (columns) indexed by S_1 with those indexed by S_3; similarly, exchanging the $n/4$ rows (columns) indexed by S_2 with those indexed by S_4;
4. reflection along the diagonal;
5. replacing each entry M_{ij} by $n^2 + 1 - M_{ij}$.

The first three properties suffice to prove that we can assume without loss of generality that the first entry $M_{1,1} = 1$. It is evident that the transformations above leave the compact definition of Franklin Magic Squares intact.

2.2 Bookkeeping

Based on the 2x2 sub-square property the square can be fixed by determining the entries on the first row and first column. For computational reasons it is more convenient to index rows and columns by $0, \ldots, n-1$, and to subtract 1 from each entry in the Franklin square, so that the entries become $0, \ldots, n^2 - 1$. Note that now the average entry value is $\nu = (n^2 - 1)/2$, instead of $(n^2 + 1)/2$. We now assume that the upper leftmost element is zero. We call this Franklin square *basic* instead of natural.

Next consider the following transformation $C(F)$ on any Franklin Magic Square F:

$$V_{ij} = C(F)_{ij} := \begin{cases} F_{ij} & \text{if } i + j \equiv 0 \text{ modulo } 2 \\ n^2 - 1 - F_{ij} & \text{if } i + j \equiv 1 \text{ modulo } 2 \end{cases}$$

which can be viewed as *complementing* entries on black positions (of the underlying chess board). Note that $F = C(V)$.

The 2x2 sub-square property of F translates into a favorable property for V, namely: $V_{i,j} + V_{i+1,j+1} - V_{i,j+1} - V_{i+1,j} = 0$, for all i, j. Based on this property, having the zero in F_{00} gives that V has the nice property that $V_{ij} = V_{i0} + V_{0j}$. Hence to generate candidate Franklin Magic Squares F we enumerate all vectors $x = (x_0, \ldots, x_{n-1})$ and $y = (y_0, \ldots, y_{n-1})$, with properties:

1. $x_0 = y_0 = 0$;
2. $x_0 < x_2 < \ldots < x_{n/2-2}$,
   $x_1 < x_3 < \ldots < x_{n/2-1}$,
   $x_{n/2} < x_{n/2+2} < \ldots < x_{n-2}$,
   $x_{n/2+1} < x_{n/2+3} < \ldots < x_{n-1}$;
3. $y_0 < y_2 < \ldots < y_{n/2-2}$,
   $y_1 < y_3 < \ldots < y_{n/2-1}$,
   $y_{n/2} < y_{n/2+2} < \ldots < y_{n-2}$,
   $y_{n/2+1} < y_{n/2+3} < \ldots < y_{n-1}$;
4. $x_0 + x_2 + \ldots + x_{n/2-2} =$
   $x_1 + x_3 + \ldots + x_{n/2-1} =$
   $x_{n/2} + x_{n/2+2} + \ldots + x_{n-2} =$
   $x_{n/2+1} + x_{n/2+3} + \ldots + x_{n-1}$;
5. $y_0 + y_2 + \ldots + y_{n/2-2} =$
   $y_1 + y_3 + \ldots + y_{n/2-1} =$
   $y_{n/2} + y_{n/2+2} + \ldots + y_{n-2} =$
   $y_{n/2+1} + y_{n/2+3} + \ldots + y_{n-1}$;
6. $x_1 < x_{n/2+1}$;
7. $y_1 < y_{n/2+1}$;
8. $\max_i y_{2i} > \max_j x_{2j}$;
9. $0 \leq y_i + x_j \leq n^2 - 1$, for all i, j;
10. the set $\{y_i + x_j | i + j \equiv 0\} \cup \{n^2 - 1 - y_i - x_j | i + j \equiv 1\}$ equals $\{0, \ldots, n^2 - 1\}$.

3 Not Finding the 12x12 Franklin Magic Square

Evidently the enumeration should be kept to a minimum by pruning the search for candidate Franklin Magic Squares as early as possible. For the 12 by 12 Franklin Square the following strategy turns out to lead to a manageable enumeration scheme.

1. generate a 3x6 sub-matrix on the 6 columns with even index and on rows indexed $0, 2, 4$;
2. extend this to a 6x6 sub-matrix on the even columns and the even rows;
3. extend to a 9x9 sub-matrix adding rows and columns indexed $1, 3$ and 5;
4. finally extend to a full 12x12 matrix

At each stage, before adding (three) more rows or (three) more columns, we update a list of candidate x_j or y_i values, given the partially filled F. Note that for instance, after the first step, possible values for y_6, y_8, y_{10} come from a limited common domain, consistent with the 3x6 sub-matrix already filled.

Proceeding in this way we generate:

831083	tuples $x_2, x_4, x_6, x_8, x_{10}$;
40467771	extensions y_4,
1473501105	extensions y_2, y_4, i.e. 3x6 sub-matrices
25663243622	extensions y_2, y_4, y_{10},
24473864360	extensions $y_2, y_4, y_6, y_8, y_{10}$, i.e. 6x6 squares,
22532519520	of which cannot be ruled out immediately;
121404978	9x9 extensions,
93083	of which might be extended to a 12x12 square.

In the end, none of these would lead to the desired Franklin Magic Square. Computation of these cases was carried out by a network of 50 computers. For this we split the work into 70 cases, corresponding with the possible settings for $x_2 \in \{1, \ldots, 70\}$. (For a higher value for x_2 we would have that $x_4 + y_{\max} \geq 72 + 73 = 145 > \max\{0, \ldots, 12^2 - 1\}$. The total computation time was approximately 160 hours.

4 A generic scheme for building Magic Squares

The above described formulation in terms of vectors x and y can also be used in a generic way to generate (Franklin) Magic Squares with the 2x2 sub-square property for arbitrary even order, with or without additional properties. To this purpose we formulate the magic square properties in terms of an equation in polynomials.

4.1 An encoding in polynomials

The polynomials we consider have coefficients 0 and 1. Let δ or $\delta(P)$ denote the degree of a polynomial P. Then for a polynomial in z, $P(z)$, of degree δ, and any number $\nu \geq \delta$, let $\overline{P}^{\nu}$ be defined by $\overline{P}^{\nu}(z) = z^{\nu} P(1/z)$. We are more or less writing P backwards, or better, we are reflecting its exponents with respect to the value $\nu/2$. If we do not mention ν we take by convention the degree of the polynomial.

Let us now associate with each (Franklin) Magic Square, given in terms of x and y, the polynomials $A(z) := \sum_{j=0}^{n} z^{x_j}$, and $B(z) := \sum_{i=0}^{n} z^{y_i}$. Let us further split these summations over odd and even indices: $A(z) = A_0(z) + A_1(z)$, with $A_k(z) := \sum_{j=0, j\equiv k}^{n} z^{x_j}$, for $k = 0, 1$; and $B(z) = B_0(z) + B_1(z)$, with $B_k(z) := \sum_{i=0, i\equiv k}^{n} z^{y_i}$, for $k = 0, 1$. A square with numbers $\{0, \ldots, n^2 - 1\}$ with the 2x2 sub-square property then satisfies the condition:

$$(A_0B_0 + A_1B_1)(z) + \overline{A_0B_1 + A_1B_0}^{n^2-1}(z) = \frac{z^{n^2} - 1}{z - 1} \tag{1}$$

which simply stipulates that all numbers are present once in the matrix. Here A_0, A_1, B_0, B_1 are polynomials with $n/2$ terms each. Solving the above system (to find the square) is possible if one restricts to certain types of solutions. One such restriction (**Type 1a**) could be to choose

$$B_0 = B_1 = \overline{B_0}^{\delta(B_0)} \tag{2}$$

which leads to the following simplification of the equation above:

$$(AB_0)(z) + \overline{AB_0}^{n^2-1}(z) = (A + \overline{A}^{n^2-1-\delta(B_0)})(z)B_0(z) = \frac{z^{n^2} - 1}{z - 1} \tag{3}$$

A variant of this approach (**Type 1b**) would be to choose

$$B_0 = B_1 \neq \overline{B_0}^{\delta(B_0)} \text{ and } A = \overline{A}^{\delta(A)} \tag{4}$$

which leads to the simplification:

$$(AB_0)(z) + \overline{AB_0}^{n^2-1}(z) = A(z)(B_0 + \overline{B_0}^{n^2-1-\delta(A)})(z) = \frac{z^{n^2}-1}{z-1} \tag{5}$$

A second approach (**Type 2**) could be to assume the existence of a number ν such that

$$A_0 = \overline{A_0}^{n^2-1-\nu}, \quad A_1 = \overline{A_1}^{n^2-1-\nu}, \quad B_0 = \overline{B_0}^{\nu}, \quad B_1 = \overline{B_1}^{\nu} \tag{6}$$

A third approach (**Type 3**) is to assume an integer ν exists such that

$$A_0 = \overline{A_1}^{n^2-1-\nu}, \quad A_1 = \overline{A_0}^{n^2-1-\nu}, \quad B_0 = \overline{B_1}^{\nu}, \quad B_1 = \overline{B_0}^{\nu} \tag{7}$$

Both (6) and (7) translate equation (1) into the simple

$$A(z)B(z) = \frac{z^{n^2}-1}{z-1} \tag{8}$$

Now, for $n = 2^q k$, with odd k, the right hand side in equation (1) can be rewritten as

$$\frac{z^{n^2}-1}{z-1} = \frac{z^{k^2 2^{2q}}-1}{z^{k2^{2q}}-1}\frac{z^{k2^{2q}}-1}{z^{2^{2q}}-1}\prod_{j=0}^{2q-1}(1+z^{2^j})$$

Note that the first two factors in this decomposition are polynomials in z with k terms each, whereas the other factors are two-term polynomials. In case $k = 1$ this decomposition is unique, but for other values there are many possible decompositions.

Using the first method to solve (1), we look for a candidate polynomial B_0, with $n/2$ terms, by selecting one of the two first factors, and $q-1$ factors from the other ones. Their product is indeed a polynomial in $n/2$ terms, and is symmetric (meaning $B_0 = \overline{B_0}^{\nu}$, for some ν). The factors not selected form a product $\Sigma(z)$ that is in fact a symmetric polynomial in $2n$ terms, and that we have to set equal to the sum $A + \overline{A}^{n^2-1-\delta(B_0)}$ by an appropriate choice for A.

For the variant we select one factor from the first two and next q factors from the second part so as to build A. The co-factor (with n terms) must then match $B_0 + \overline{B_0}^{n^2-1-\delta(A)}$ for an appropriate choice of B_0.

When using the second or third method, we may define B say, by taking one of the two first factors, and adding q factors from the other ones. Their product is then a symmetric polynomial in n terms, which can further be split into B_0 and B_1. The remaining factors are used to build A_0 and A_1.

In the remainder of the paper we show how to construct various types of Franklin Magic Squares. We first formulate how additional requirements on the constructed squares translate into conditions on the vectors x and y, and hence on the polynomials A_0, A_1, B_0, B_1. Define $X_{ik} = \sum_{j\equiv i, \lfloor 2j/n \rfloor = k} x_j$ for $i, k \in \{0, 1\}$. Similarly, let $Y_{ik} = \sum_{j\equiv i, \lfloor 2j/n \rfloor = k} y_j$.

magic row sum $X_{00} + X_{01} = X_{10} + X_{11}$ or, equivalently, the sum of exponents in A_0 equals the sum of exponents in A_1;

magic column sum $Y_{00} + Y_{01} = Y_{10} + Y_{11}$ or, equivalently, the sum of exponents in B_0 equals the sum of exponents in B_1;

magic sum on horizontal bent diagonals $X_{00} = X_{11}$ and $X_{01} = X_{10}$, or, equivalently, $A_0 = A_{00} + A_{01}$, $A_1 = A_{10} + A_{11}$ is a split into four polynomials of $n/4$ terms each with exponents in A_{00} (A_{10}) adding up to the same as those in A_{11} (A_{01}, respectively);

magic sum on vertical bent diagonals $Y_{00} = Y_{11}$ and $Y_{01} = Y_{10}$, or, equivalently, $B_0 = B_{00} + B_{01}$, $B_1 = B_{10} + B_{11}$ is a split into four polynomials of $n/4$ terms each with exponents in B_{00} (B_{10}) adding up to the same as those in B_{11} (B_{01}, respectively);

half the magic sum in first and second half row $X_{00} = X_{10}$ and $X_{01} = X_{11}$, or, equivalently, there is a split of A, as above, with exponents in A_{00} summing to the same as those in A_{10}, and exponents in A_{01} summing to the same as those in A_{11};

half the magic sum in first and second half column $Y_{00} = Y_{10}$ and $Y_{01} = Y_{11}$, or, equivalently, there is a split of B, as above, with exponents in B_{00} summing to the same as those in B_{10}, and exponents in B_{01} summing to the same as those in B_{11};

pan-diagonal magic sum $X_{00} + X_{01} + X_{10} + X_{11} + Y_{00} + Y_{01} + Y_{10} + Y_{11} = n(n^2 - 1)/2$, or equivalently, exponents in A and B add up to the magic sum;

most-perfect This means: complementary entries lie on the same diagonal, $n/2$ positions apart. That is, $M_{i,j} + M_{i+n/2,j+n/2} = (n^2 + 1)$, for all i, j. We then have $x_j + x_{j+n/2} + y_i + y_{i+n/2} = n^2 - 1$, for all i, j, implying that $x_j + x_{j+n/2} = \delta(A)$, for all $j < n/2$, and $y_i + y_{i+n/2} = \delta(B)$, for all $i < n/2$, and each of A_0, A_1, B_0, B_1 must be symmetric;

four-on-a-row This means: blocks of 4 consecutive entries partitioning a row (or column) each have magic entry sum. In other words, $M_{i,4k+1} + M_{i,4k+2} + M_{i,4k+3} + M_{i,4k+4} = 2(n^2+1)$, for all i, k. Then $x_{4j} + x_{4j+2} = x_{4j+1} + x_{4j+3}$, for all j, implying that pairs of exponents in A_{00} match with pairs of exponents in A_{10} having the same sum, etcetera.

4.2 Simple Magic Squares

4.2.1 Method 1a

If we pose no further restrictions, then for each symmetric polynomial $\Sigma(z)$ of $2n$ terms we can easily find 2^n different solutions $A(z)$ as follows: for the n smallest powers z^j in Σ we have that z^{N-j} is in Σ as well, where $N = \delta(\Sigma)$. For each j, select one of $\{j, N - j\}$ to be in the set of powers of A. For instance

$$1 + z^2 + z^3 + z^6 + z^9 + z^{12} + z^{13} + z^{15} = (1 + z^3 + z^6 + z^{13}) + \overline{(1 + z^3 + z^6 + z^{13})}^{15}$$

Now A and B will lead to a square matrix of numbers $\{0, \ldots, n^2 - 1\}$ satisfying the 2x2 sub-square property. Each column will have fixed sum $n(n^2 - 1)/2$, for the simple reason that B_0 and B_1 are equal.

In order to have fixed row sums as well, we should be able to split $A(z) = A_0(z) + A_1(z)$, with exponents in A_0 adding to the same sum as the exponents in A_1. This can be done in general as follows. Let $1+z^{2^j}$ be a factor of Σ, that is $\Sigma(z) = (1+z^{2^j})\Omega(z)$, where Ω is symmetric as well, and with (even) n terms. Pair the terms in $\Omega(z)$ with matching exponents $(z^t + z^{\delta(\Omega)-t})$. Taking $A(z) = \Omega(z)$ gives $\overline{A}^{\delta(\Omega)+2^j}(z) = z^{2^j}A(z)$. Now split A into A_0 and A_1 by taking for A_0 half of the $n/2$ pairs in $\Omega(z)$, and for A_1 the remaining $n/4$ pairs. As each pair contributes $\delta(\Omega)$ to the sum of exponents, the split will be balanced. So the exponents in A_0 will add up to the same sum as those in A_1, and hence the resulting matrix will have constant row sums.

4.2.2 Method 1b

In order to find Type 1b squares with fixed row and column sum, we have to be able to split the n-term polynomial A into two parts of equal exponent sum. This is easily achieved by the method described above: form pairs of matching terms $z^j, z^{\delta(A)-j}$, and divide these pairs over two groups of equal size. If n is a multiple of four, such a split is possible in $\binom{n/2}{n/4}$ ways, for any given A.

In order to find a matching B_0 it suffices to pair matching terms $z^j, z^{n^2-1-\delta(A)-j}$ and select one of each pair as an element of B_0. There are $2^{n/2}$ possible B_0's, for a given A. By definition B_0 and B_1 have the same sum of exponents.

4.2.3 Method 2

After decomposing $\frac{z^{n^2}-1}{z-1} = A(z)B(z)$ where both A and B have n terms, we have lots of ways to split A into parts A_0, A_1 with $A_i = \overline{A_i}^{\delta(A)}$ by pairing the jth lowest term with the jth highest term in A, and then assign half of these pairs to A_0 and the other half to A_1. Similar for B. There are $\binom{n/2}{n/4}$ ways to generate A_0 and there are $\binom{n/2}{n/4}$ ways to generate B_0 (for fixed A and B). We only need n to be a multiple of 4.

Note that by keeping matching exponents close together method 2 as described above generates 4x4 blocks that have the property that each row, each column and each (broken) diagonal has magic sum $2(n^2+1)$. So if we apply this method to generate squares of order $4k$, only caring about 4 fields on-a-row having fixed sum $2(n^2+1)$, we get squares that have the property that each 8x8 sub-square has all the Franklin Magic Square properties as far as row, column and bent-diagonal sums are concerned, with average entry value equal to $(n^2+1)/2$. For instance take $n = 12$, $A(z) = (1+z^{48}+z^{96})(1+z^8)(1+z^4)$, and $B(z) = (1+z^{16}+z^{32})(1+z^2)(1+z)$. With $x = (0, 4, 108, 104, 8, 12, 100, 96, 48, 52, 60, 56)$ and $y = (0, 1, 35, 34, 2, 3, 33, 32, 16, 17, 19, 18)$ we obtain the square $M_{12.2}$ given in Figure 1. It contains four 8x8 subsquares, aligned with the 4x4 block structure, with all the Franklin Magic Square properties (except for containing 64 consecutive numbers). Notice that each 4x4 block in the structure is most-perfect in the sense that its 2x2 subsquares are one another's complement. Further observe that in this example every 4x4 block with upper left entry in an odd row and an odd column has magic row and column sum! This can be enforced in general by building the x-vector in strips of four with values $j, j+\alpha, N-j, N-j-\alpha$, for some fixed α, and similarly build the y-vector in strips of four of value $j, j+\beta, N'-j, N'-j-\beta$, for some fixed β. Here $N = \delta(A)$ and $N' = \delta(B)$. The features are highlighted in bold font.

$$M_{12.2} = \begin{array}{|rrrr|rrrr|rrrr|}
\hline
\mathbf{1} & \mathbf{140} & 109 & 40 & 9 & 132 & 101 & 48 & 49 & 92 & 61 & 88 \\
\mathbf{143} & \mathbf{6} & 35 & 106 & 135 & 14 & 43 & 98 & 95 & 54 & 83 & 58 \\
36 & 105 & \mathbf{144} & \mathbf{5} & 44 & 97 & 136 & 13 & 84 & 57 & 96 & 53 \\
110 & 39 & \mathbf{2} & \mathbf{139} & 102 & 47 & 10 & 131 & 62 & 87 & 50 & 91 \\
\hline
3 & 138 & 111 & 38 & 11 & 130 & 103 & 46 & 51 & 90 & 63 & 86 \\
141 & 8 & 33 & 108 & 133 & 16 & 41 & 100 & 93 & 56 & 81 & 60 \\
34 & 107 & \mathbf{142} & \mathbf{7} & \mathbf{42} & \mathbf{99} & 134 & 15 & 82 & 59 & 94 & 55 \\
112 & 37 & \mathbf{4} & \mathbf{137} & \mathbf{104} & \mathbf{45} & 12 & 129 & 64 & 85 & 52 & 89 \\
\hline
17 & 124 & \mathbf{125} & \mathbf{24} & \mathbf{25} & \mathbf{116} & 117 & 32 & 65 & 76 & 77 & 72 \\
127 & 22 & \mathbf{19} & \mathbf{122} & \mathbf{119} & \mathbf{30} & 27 & 114 & 79 & 70 & 67 & 74 \\
20 & 121 & 128 & 21 & 28 & 113 & 120 & 29 & 68 & 73 & 80 & 69 \\
126 & 23 & 18 & 123 & 118 & 31 & 26 & 115 & 78 & 71 & 66 & 75 \\
\hline
\end{array}$$

Figure 1: Block structure with 4x4 most-perfect magic subsquares

4.2.4 Method 3

After decomposing $\frac{z^{n^2}-1}{z-1} = A(z)B(z)$ where both A and B have n terms, we have lots of ways to split A into parts A_0, A_1 with $A_0 = \overline{A_1}$. However we need to enforce equal sums of exponents. By extracting a factor $(1+z^\alpha)(1+z^\beta)$ from $A(z)$: $A(z) = (1+z^\alpha)(1+z^\beta)\Omega(z)$, we can take care for this. Match terms z^j and z^{N-j} in $\Omega(z)$, where $N = \delta(\Omega)$, and write $(1+z^\alpha)(1+z^\beta)(z^j+z^{N-j}) = (z^j + z^{\alpha+N-j} + z^{\beta+N-j} + z^{\alpha+\beta+j}) + (z^{N-j} + z^{\alpha+j} + z^{\beta+j} + z^{\alpha+\beta+N-j})$. Both four-tuples have exponent sum $2(N+\alpha+\beta)$. Assign one 4-tuple to A_0, and the other to A_1. There are $2^{n/8}$ such assignments, for fixed α, β, and there are many ways to choose α and β, for fixed A.

Similarly, for B we find several ways to come up with a proper partition into B_0 and B_1.

4.3 Magic Squares with bent-diagonals with magic sum

As indicated before, the properties of 2x2 subsquares having fixed sum, and rows and columns having fixed magic sum, lead to the equivalence of the bent-diagonal property with the condition that odd positions in the first half and even positions in the second half of the first row add up to half the magic sum. This in terms of x means $x_0+x_2+\ldots+x_{n/2-2} = x_{n/2+1}+\cdots+x_{n-1}$, and in terms of our polynomials this means that A_0 and A_1 must have a subset of $n/4$ terms each with the same exponent sum.

If n is a multiple of 8, the above construction of A by method 1a or 1b already provides such a decomposition of A. And for $B = B_0 + B_1$ it is easy to distribute the terms of B_0 and B_1 in a symmetric way. Simply take $y_i = y_{n-1-i}$, for all i (we had $B_0 = B_1$).

If n is a multiple of 4, method 2 applied in the previous section yields pairs of terms each with the same exponent sum. Keeping these pairs adjacent (i.e. on positions i and $i+2$) and in the same half (i.e. $i+2 < n/2$ or $i \geq n/2$) yields x and y vectors with the right properties. This requires n to be a multiple of 8.

As method 3 generates 4-tuples of equal exponent sum we can nicely distribute such 4-tuples provided n is a multiple of 16. Simply keep 4-tuples adjacent (on positions $i, i+2, i+4, i+6$) and on the same half.

4.4 Magic Squares with half rows having half the magic sum

If we insist on the property of having half rows with half the magic sum, and not necessarily having the bent-row property, we can do the same as in the previous subsection. Indeed, in order to have half the magic sum in the first half of the first row it suffices to have a subset of $n/4$ terms in A_0 and a subset of $n/4$ terms in A_1 having the same sum of exponents. But this was exactly the same condition we needed for having bent-diagonals with magic sum.

By interchanging the columns $1, 3, \ldots, n/2-1$ with the set of columns $n/2+1, n/2+3, \ldots, n-1$ a magic square with magic sum on horizontal bent diagonals transforms into one with half the magic sum on half rows, and vice versa. Similarly for vertical bent diagonals and half columns with half the magic sum. Hence the construction for magic squares with bent diagonals having magic sum, can be used to generate magic squares with half the magic sum on half rows and half columns.

5 Full Franklin Magic Squares of order $8k$

We can have both half rows and columns with half the magic sum, and bent-diagonals with magic sum, if both A and B can be split into four parts each with $n/4$ terms, such that the subsets of A have the same sum of exponents, and the subsets of B have the same exponent sum. As an exponent cannot appear four times a requirement is that n is at least 8.

In this section we discuss general construction methods for Franklin Magic Squares given that the order is a multiple of 8, with and without special features such as pan-diagonality and perfectness.

5.1 Regular constructions of Franklin Magic Squares

5.1.1 Method 1a

By construction along method 1a A already admits the partition into four parts of equal size and equal exponent sum. For B we merely have to pair each z^j in $B_0(z)$ with $z^{\delta(B_0)-j}$. Again if n is a multiple of 8, it is then possible to split B_0 into two sets of terms with $n/8$ pairs each.

For example, for $n=8$ one can take

$$\frac{z^{64}-1}{z-1} = \underbrace{(z^{32}+1)(z^{16}+1)(z^{8}+1)}_{A(z)}(z^4+1)\underbrace{(z^2+1)(z+1)}_{B_0(z)}$$

which yields $A(z) = (1+z^{56}) + (z^8+z^{48}) + (z^{16}+z^{40}) + (z^{24}+z^{32})$, and $B_0(z) = (1+z^3) + (z^1+z^2)$. Via vectors $x = (0, 16, 56, 40, 8, 24, 48, 32)$, and $y = (0, 2, 3, 1, 1, 3, 2, 0)$ we obtain the 8x8 squares

$$M_{1a} = \begin{array}{|cccccccc|}\hline 1 & 48 & 57 & 24 & 9 & 40 & 49 & 32 \\ 62 & 19 & 6 & 43 & 54 & 27 & 14 & 35 \\ 4 & 45 & 60 & 21 & 12 & 37 & 52 & 29 \\ 63 & 18 & 7 & 42 & 55 & 26 & 15 & 34 \\ 2 & 47 & 58 & 23 & 10 & 39 & 50 & 31 \\ 61 & 20 & 5 & 44 & 53 & 28 & 13 & 36 \\ 3 & 46 & 59 & 22 & 11 & 38 & 51 & 30 \\ 64 & 17 & 8 & 41 & 56 & 25 & 16 & 33 \\ \hline\end{array} \qquad M_{1b} = \begin{array}{|cccccccc|}\hline 1 & 48 & 57 & 24 & 9 & 40 & 49 & 32 \\ 60 & 21 & 4 & 45 & 52 & 29 & 12 & 37 \\ 6 & 43 & 62 & 19 & 14 & 35 & 54 & 27 \\ 63 & 18 & 7 & 42 & 55 & 26 & 15 & 34 \\ 2 & 47 & 58 & 23 & 10 & 39 & 50 & 31 \\ 59 & 22 & 3 & 46 & 51 & 30 & 11 & 38 \\ 5 & 44 & 61 & 20 & 13 & 36 & 53 & 28 \\ 64 & 17 & 8 & 41 & 56 & 25 & 16 & 33 \\ \hline\end{array}$$

Figure 2: Franklin Magic Squares obtained by methods 1a and 1b

$$V = \begin{array}{|cccccccc|}\hline 0 & 16 & 56 & 40 & 8 & 24 & 48 & 32 \\ 2 & 18 & 58 & 42 & 10 & 26 & 50 & 34 \\ 3 & 19 & 59 & 43 & 11 & 27 & 51 & 35 \\ 1 & 17 & 57 & 41 & 9 & 25 & 49 & 33 \\ 1 & 17 & 57 & 41 & 9 & 25 & 49 & 33 \\ 3 & 19 & 59 & 43 & 11 & 27 & 51 & 35 \\ 2 & 18 & 58 & 42 & 10 & 26 & 50 & 34 \\ 0 & 16 & 56 & 40 & 8 & 24 & 48 & 32 \\ \hline\end{array} \longrightarrow F = \begin{array}{|cccccccc|}\hline 0 & 47 & 56 & 23 & 8 & 39 & 48 & 31 \\ 61 & 18 & 5 & 42 & 53 & 26 & 13 & 34 \\ 3 & 44 & 59 & 20 & 11 & 36 & 51 & 28 \\ 62 & 17 & 6 & 41 & 54 & 25 & 14 & 33 \\ 1 & 46 & 57 & 22 & 9 & 38 & 49 & 30 \\ 60 & 19 & 4 & 43 & 52 & 27 & 12 & 35 \\ 2 & 45 & 58 & 21 & 10 & 37 & 50 & 29 \\ 63 & 16 & 7 & 40 & 55 & 24 & 15 & 32 \\ \hline\end{array}$$

and finally we obtain a square M_{1a} given in Figure 2.

5.1.2 Method 1b

When we apply method 1b we again have to able to split A into four parts with equal exponent sum, and B_0 into two parts with equal exponent sum. The first part is easy because A is symmetric and if n is a multiple of 8 we can easily create $n/2$ pairs and partition them over 4 groups. As B_0 is not symmetric in all cases we have to enforce this by defining $(B_0 + \overline{B_0}^{n^2-1-\delta(A)})(z) = (1 + z^\alpha)\Omega(z)$, and take $B_0 = \Omega$. Match complementary terms and split the set of pairs in two.

For an example of method 1b let us consider $A(z) = (z^{32}+1)(z^{16}+1)(z^8+1)$ as above, and $(B_0+\overline{B_0}^7)(z) = (z^4+1)(z^2+1)(z+1) = (z^2+1)B_0(z)$, with $B_0(z) = B_1(z) = (1+z^5)+(z^1+z^4)$. With vectors $x = (0, 16, 56, 40, 8, 24, 48, 32)$ and $y = (0, 4, 5, 1, 1, 5, 4, 0)$ this leads to matrix M_{1b} given in Figure 2.

Note that both method 1a and 1b lead to symmetry along the horizontal axis: each entry f mirrors its complement $n^2 + 1 - f$.

5.1.3 Method 2

Application of method 2 immediately generates A and B consisting of pairs of terms with sums of exponents equal to $\delta(A)$ and $\delta(B)$, respectively. As a side-result, all matrices obtained

$M_2 =$

1	48	57	24	9	40	49	32
62	19	6	43	54	27	14	35
8	41	64	17	16	33	56	25
59	22	3	46	51	30	11	38
2	47	58	23	10	39	50	31
61	20	5	44	53	28	13	36
7	42	63	18	15	34	55	26
60	21	4	45	52	29	12	37

Figure 3: Franklin Magic Square obtained by method 2

in this way will be magic squares that are pan-diagonal. Further, keeping the pairs adjacent (i.e. on positions i and $i+2$) and on the same half (either $i+2 < n/2$ or $i \geq n/2$) yields x and y vectors with the right properties, in particular they yield matrices with the bent-diagonal property. For the latter to be true, n must be a multiple of 8.

For an example of method 2 let us consider $A(z) = (z^{32}+1)\mathbf{(z^{16}+1)}(z^8+1)$ as above, and $B(z) = (z^4+1)\mathbf{(z^2+1)}(z+1)$, with $B_0(z) = (1+z^7)+(z^1+z^6)$, and $B_1(z) = (z^2+z^5)+(z^3+z^4)$. With vectors $x = (0, 16, 56, 40, 8, 24, 48, 32)$ and $y = (0, 2, 7, 5, 1, 3, 6, 4)$ this leads to matrix M_2 given in Figure 3. Again, notice the most-perfectness of the 4x4 blocks.

5.1.4 Method 3

Application of method 3 yields 4-tuples of the same exponent sums equal to $2\delta(A)$ and $2\delta(B)$, hence for multiples of 16 it works. As an example let us take $A(z) = (z^{128}+1)(z^{32}+1)(z^{16}+1)(z^8+1)$, and $B = (z^{64}+1)(z^4+1)(z^2+1)(z+1)$. For splitting A we take $\alpha = 128$, $\beta = 32$, $N = 24$ and we get $A_{00}(z) = z^0+z^{152}+z^{56}+z^{160}$, $A_{10}(z) = z^{24}+z^{128}+z^{32}+z^{184}$, $A_{01}(z) = z^8+z^{144}+z^{48}+z^{168}$, $A_{11}(z) = z^{16}+z^{136}+z^{40}+z^{176}$. For splitting B we have $\alpha = 64$, $\beta = 4$, $N = 3$ yielding $B_{00}(z) = z^0+z^{67}+z^7+z^{68}$, $B_{10}(z) = z^3+z^{64}+z^4+z^{71}$, $B_{01}(z) = z^1+z^{66}+z^6+z^{69}$, $B_{11}(z) = z^2+z^{65}+z^5+z^{70}$. With vectors $x = (0, 24, 152, 128, 56, 32, 160, 184, 8, 16, 144, 136, 48, 40, 168, 176)$, and $y = (0, 3, 67, 64, 7, 4, 68, 71, 1, 2, 66, 65, 6, 5, 69, 70)$ this yields M_3 as given in Figure 4.

Notice that each 8x8 quadrant is *rotationally anti-symmetric*: rotating the quadrant by 180 degrees maps each entry on its complement.

5.2 Pan-diagonal Franklin Magic Squares

We may also want to enforce squares with diagonals having the magic sum. Then in addition to the previous conditions we have to restrict ourselves to polynomials A and B each splittable in four subsets of equal exponent sum, such that the sum of exponents of A_0 and B_0 add up to the desired value $n(n^2-1)/4$. Application of methods 2 and 3 directly leads to pan-diagonal Franklin Magic Squares, as by construction the average values of the x_j and y_i add up to $(n^2-1)/2$.

$M_3 =$

1	232	153	128	57	224	161	72	9	240	145	120	49	216	169	80
253	28	101	132	197	36	93	188	245	20	109	140	205	44	85	180
68	165	220	61	124	157	228	5	76	173	212	53	116	149	236	13
192	89	40	193	136	97	32	249	184	81	48	201	144	105	24	241
8	225	160	121	64	217	168	65	16	233	152	113	56	209	176	73
252	29	100	133	196	37	92	189	244	21	108	141	204	45	84	181
69	164	221	60	125	156	229	4	77	172	213	52	117	148	237	12
185	96	33	200	129	104	25	256	177	88	41	208	137	112	17	248
2	231	154	127	58	223	162	71	10	239	146	119	50	215	170	79
254	27	102	131	198	35	94	187	246	19	110	139	206	43	86	179
67	166	219	62	123	158	227	6	75	174	211	54	115	150	235	14
191	90	39	194	135	98	31	250	183	82	47	202	143	106	23	242
7	226	159	122	63	218	167	66	15	234	151	114	55	210	175	74
251	30	99	134	195	38	91	190	243	22	107	142	203	46	83	182
70	163	222	59	126	155	230	3	78	171	214	51	118	147	238	11
186	95	34	199	130	103	26	255	178	87	42	207	138	111	18	247

Figure 4: Franklin Magic Square by method 3

For methods 1a and 1b we can enforce this feature in various ways.

5.2.1 Method 1a

Consider in the decomposition of $\frac{z^{n^2}-1}{z-1}$ a factor product of the form

$$W_{\alpha,\beta}(z) = (1+z^\alpha)(1+z^\beta) = 1 + z^\alpha + z^\beta + z^{\alpha+\beta}$$

We choose $W_{\alpha,\beta}$ to be a factor of $A + \overline{A}^{n^2-1-\delta(B_0)}$. In $(A + \overline{A}^{n^2-1-\delta(B_0)})(z)/W_{\alpha,\beta}(z)$ let us pair up terms z^j and z^{N-j}, where N is the degree of the co-factor.

Notice that $W_{\alpha,\beta}(z)(z^j + z^{N-j}) = (z^j + z^{\alpha+N-j} + z^{\beta+N-j} + z^{\alpha+\beta+j}) + (z^{N-j} + z^{\alpha+j} + z^{\beta+j} + z^{\alpha+\beta+N-j})$. The first four terms have exponent sum $2N+2\alpha+2\beta$, and the same holds for the last four terms. Hence the average exponent value is $(N+\alpha+\beta)/2$, which is half the degree of $A+\overline{A}^{n^2-1-\delta(B_0)}$. Note that the two parts are each others complement (with respect to power $\nu = N+\alpha+\beta$).

One further observation is that in both 4-tuples, the first two terms have exponents adding up to $N+\alpha$, whereas the second pair has exponent sum $N+\alpha+2\beta$. Assign one of the two parts to A. This split is actually already possible for n being a multiple of four. If n is a multiple of 16, the aforementioned method allows us to generate polynomials A that can be split up in four groups with $n/4$ terms each, such that within each group the average exponent equals $(n^2-1-\delta(B_0))/2$. Now, together with averaged exponents in B_0 this leads to Franklin Magic Squares that have the additional property that all diagonals have the magic sum, and all half diagonals (i.e. diagonals within each quadrant) have half the magic sum.

Working out the above approach for $n = 16$ yields 40320 different pan-diagonal Franklin Magic Squares the first of which is generated by:

x	0	96	225	129	226	130	3	99	160	32	65	193	66	194	163	35
y	0	4	28	24	8	12	20	16	16	20	12	8	24	28	4	0
A_0	0	225	226	3	160	65	66	163								
A_1	96	129	130	99	32	193	194	35								
B_0	0	28	8	20	16	12	24	4								
B_1	4	24	12	16	20	8	28	0								

which yields a square M_{pd1a} given in Figure 5.

The square contains all numbers from 1 to 256, with rows, columns and diagonals each summing to 2056; with half rows, half columns and half main and back diagonal summing to 1028; with bent-diagonals summing to 2056, and with each 2x2 square having sum 514. Each four-on-a-row has sum 514. The four sub-matrices are magic themselves, with constant row, column and diagonal sums, including parallels of the diagonals and back diagonals. The matrix is anti-symmetric along the horizontal line of symmetry, opposite entries add up to 257.

5.2.2 Method 1b

In this case we have to be able to split B_0 into two parts with equal exponent sum and we like to retain the horizontal axis of symmetry. We borrow from the trick we applied for method 1a, and identify a factorization of $(B_0 + \overline{B_0}^{n^2-1-\delta(A)})(z) = W_{\alpha,\beta}(z)\Omega(z)$. We pair up terms

$M_{pd1a} =$

1	160	226	127	227	126	4	157	161	224	66	63	67	62	164	221
252	101	27	134	26	135	249	104	92	37	187	198	186	199	89	40
29	132	254	99	255	98	32	129	189	196	94	35	95	34	192	193
232	121	7	154	6	155	229	124	72	57	167	218	166	219	69	60
9	152	234	119	235	118	12	149	169	216	74	55	75	54	172	213
244	109	19	142	18	143	241	112	84	45	179	206	178	207	81	48
21	140	246	107	247	106	24	137	181	204	86	43	87	42	184	201
240	113	15	146	14	147	237	116	80	49	175	210	174	211	77	52
17	144	242	111	243	110	20	141	177	208	82	47	83	46	180	205
236	117	11	150	10	151	233	120	76	53	171	214	170	215	73	56
13	148	238	115	239	114	16	145	173	212	78	51	79	50	176	209
248	105	23	138	22	139	245	108	88	41	183	202	182	203	85	44
25	136	250	103	251	102	28	133	185	200	90	39	91	38	188	197
228	125	3	158	2	159	225	128	68	61	163	222	162	223	65	64
5	156	230	123	231	122	8	153	165	220	70	59	71	58	168	217
256	97	31	130	30	131	253	100	96	33	191	194	190	195	93	36

Figure 5: Pan-diagonal Franklin Magic Square by method 1a

z^j and z^{N-j} in $\Omega(z)$, where $N = \delta(\Omega)$.
As before, rewrite $W_{\alpha,\beta}(z)(z^j + z^{N-j}) = (z^j + z^{\alpha+N-j} + z^{\beta+N-j} + z^{\alpha+\beta+j}) + (z^{N-j} + z^{\alpha+j} + z^{\beta+j} + z^{\alpha+\beta+N-j})$. The first four terms have exponent sum $2N+2\alpha+2\beta$, and the same holds for the last four terms. Hence the average exponent value is $(N + \alpha + \beta)/2$, which is half the degree of $B_0 + \overline{B_0}^{n^2-1-\delta(A)}$. Note that the two parts are each others complement (with respect to power $\nu = N + \alpha + \beta$). Select one of the four-tuples to be a part of B_0.

If n is a multiple of 16, $\Omega(z)$ contains an even number of matched pairs z^j, z^{N-j}. The four-tuples destined for B_{00} remain as they are, the four-tuples for B_{01} should be reversed in order, that is, rewritten as $(z^{\alpha+\beta+j} + z^{\beta+N-j} + z^{\alpha+N-j} + z^j)$. By doing so the pairs adjacent terms in B_0 will nicely match with the pairs of adjacent terms in B_1. Again we define $y_{n-1-i} = y_i$, for each even i.
Application of method 1b is illustrated by the following example. Let us take $A(z) = (z^{128} + 1)(z^{32} + 1)(z^{16} + 1)(z^8 + 1)$, and $B_0 + \overline{B_0}^{71} = (z^{64} + 1)(z^4 + 1)(z^2 + 1)(z + 1)$. For splitting A into four parts we simply take matching pairs z^j, z^{184-j} and distribute these pairs evenly. We may obtain $A_{00}(z) = z^0 + z^{184} + z^8 + z^{176}$, $A_{10}(z) = z^{16} + z^{168} + z^{24} + z^{160}$, $A_{01}(z) = z^{32} + z^{152} + z^{40} + z^{144}$, $A_{11}(z) = z^{48} + z^{136} + z^{56} + z^{128}$. To obtain B_0 let us take $\alpha = 64$, $\beta = 4$, $N = 3$. We may get $B_{00}(z) = z^0 + z^{67} + z^7 + z^{68}$, $B_{10}(z) = z^1 + z^{66} + z^6 + z^{69}$, $B_{01}(z) = z^{69} + z^6 + z^{66} + z^1$, $B_{11}(z) = z^{68} + z^7 + z^{67} + z^0$. Now A has average exponent 92 and B has average exponent 71/2 which sums up to $255/2 = (n^2 - 1)/2$. With vectors $x = (0, 16, 184, 168, 8, 24, 176, 160, 32, 48, 152, 136, 40, 56, 144, 128)$ and $y = (0, 1, 67, 66, 7, 6, 68, 69, 69, 68, 6, 7, 66, 67, 1, 0)$ we obtain matrix M_{pd1b} in Figure 6.

5.3 Most-perfect Squares

Sometimes we like to have yet another even stronger requirement for symmetry: diagonals should be composed of pairs of complementary integers, at distance $n/2$. Complementary integers are pairs of entries with sum $(n^2 + 1)$. Being $n/2$ apart (which is even) they must match with exponents x_j, $x_{j+n/2}$ both in A_0 or both in A_1. Hence, methods 1a, 1b and 3 cannot yield such solutions. Method 2 does create solutions that have the right property. It is a matter of ordering the coefficients in x and y respectively in the right way so as to have $x_j + x_{j+n/2} = \delta(A)$, for all $j < n/2$ and $y_i + y_{i+n/2} = \delta(B)$, for all $i < n/2$. For a given A, as before, write $A(z) = (1 + z^\alpha)(1 + z^\beta)\Omega(z)$, and consider matching terms z^j and z^{N-j}.

Now we rewrite $(1 + z^\alpha)(1 + z^\beta)(z^j + z^{N-j}) = (z^j + z^{\alpha+N-j} + z^{\beta+N-j} + z^{\alpha+\beta+j}) + (z^{\alpha+\beta+N-j} + z^{\beta+j} + z^{\alpha+j} + z^{N-j})$. Now the order in the second 4-term has been rearranged such that complementary terms can be offset in the x-vector by $n/2$ positions. The first 4-tuples are used for building the polynomials A_{00} and A_{10}, the second 4-tuples are used for A_{01} and A_{11}.

For $n = 16$ this approach leads to 1260 different most-perfect Franklin Magic Squares, with the additional property of four-on-a-row. The first in the series was generated by A, B, x and y given by

x	0	64	208	144	224	160	48	112	240	176	32	96	16	80	192	128
y	0	4	13	9	14	10	3	7	15	11	2	6	1	5	12	8
A_0	0	208	224	48	240	32	16	192								
A_1	64	144	160	112	176	96	80	128								
B_0	0	13	14	3	15	2	1	12								
B_1	4	9	10	7	11	6	5	8								

$M_{pd1b} =$

1	240	185	88	9	232	177	96	33	208	153	120	41	200	145	128
255	18	71	170	247	26	79	162	223	50	103	138	215	58	111	130
68	173	252	21	76	165	244	29	100	141	220	53	108	133	212	61
190	83	6	235	182	91	14	227	158	115	38	203	150	123	46	195
8	233	192	81	16	225	184	89	40	201	160	113	48	193	152	121
250	23	66	175	242	31	74	167	218	55	98	143	210	63	106	135
69	172	253	20	77	164	245	28	101	140	221	52	109	132	213	60
187	86	3	238	179	94	11	230	155	118	35	206	147	126	43	198
70	171	254	19	78	163	246	27	102	139	222	51	110	131	214	59
188	85	4	237	180	93	12	229	156	117	36	205	148	125	44	197
7	234	191	82	15	226	183	90	39	202	159	114	47	194	151	122
249	24	65	176	241	32	73	168	217	56	97	144	209	64	105	136
67	174	251	22	75	166	243	30	99	142	219	54	107	134	211	62
189	84	5	236	181	92	13	228	157	116	37	204	149	124	45	196
2	239	186	87	10	231	178	95	34	207	154	119	42	199	146	127
256	17	72	169	248	25	80	161	224	49	104	137	216	57	112	129

Figure 6: Pan-diagonal Franklin Magic Square obtained with method 1b

$M_{pf2} =$

1	192	209	112	225	96	49	144	241	80	33	160	17	176	193	128
252	69	44	149	28	165	204	117	12	181	220	101	236	85	60	133
14	179	222	99	238	83	62	131	254	67	46	147	30	163	206	115
247	74	39	154	23	170	199	122	7	186	215	106	231	90	55	138
15	178	223	98	239	82	63	130	255	66	47	146	31	162	207	114
246	75	38	155	22	171	198	123	6	187	214	107	230	91	54	139
4	189	212	109	228	93	52	141	244	77	36	157	20	173	196	125
249	72	41	152	25	168	201	120	9	184	217	104	233	88	57	136
16	177	224	97	240	81	64	129	256	65	48	145	32	161	208	113
245	76	37	156	21	172	197	124	5	188	213	108	229	92	53	140
3	190	211	110	227	94	51	142	243	78	35	158	19	174	195	126
250	71	42	151	26	167	202	119	10	183	218	103	234	87	58	135
2	191	210	111	226	95	50	143	242	79	34	159	18	175	194	127
251	70	43	150	27	166	203	118	11	182	219	102	235	86	59	134
13	180	221	100	237	84	61	132	253	68	45	148	29	164	205	116
248	73	40	153	24	169	200	121	8	185	216	105	232	89	56	137

Figure 7: Most-perfect Franklin Magic Square, by method 2

and the resulting square M_{pf2} is given in Figure 7.

6 Franklin Magic Squares of order 20 and higher

In section 3 it was shown that no 12 by 12 Franklin Magic Square exists. It turns out that this is a unique exception. Below we show how to construct a Franklin Magic Square of order $20 + 8k$, for $k \geq 0$. We first construct two squares of order 20.

6.1 Franklin Magic Squares of order 20

Using method 1a we aim for a polynomial A of 20 terms, and a polynomial B_0 of 10 terms, such that $(A + \overline{A}^{399-\delta(B_0)})(z)B_0(z) = \frac{z^{400}-1}{z-1}$. We need that A can be split into four parts of five terms with equal exponent sum, and B_0 must be split into two parts of 5 terms each, again with equal exponent sum.

A candidate solution for B_0 is of the form $(1 + z^\gamma + z^{2\gamma} + z^{3\gamma} + z^{4\gamma})(1 + z^{10\gamma})$ which can be split into $(z^0 + z^\gamma + z^{10\gamma} + z^{11\gamma} + z^{13\gamma}) + (z^{2\gamma} + z^{3\gamma} + z^{4\gamma} + z^{12\gamma} + z^{14\gamma})$. Each part has exponent sum 35γ.

A candidate solution for A is derived from the general form $(A + \overline{A}^\nu)(z) = (1 + z^\alpha)(1 + z^\beta + z^{2\beta} + \ldots + z^{19\beta})$. One possible solution is $A_{\alpha,\beta}(z) := (z^0 + z^\beta + z^{7\beta} + z^{16\beta+\alpha} + z^{17\beta}) + (z^\alpha + z^{4\beta} + z^{8\beta} + z^{11\beta} + z^{18\beta}) + (z^{4\beta+\alpha} + z^{5\beta} + z^{9\beta} + z^{10\beta} + z^{13\beta}) + (z^{2\beta} + z^{5\beta+\alpha} + z^{6\beta} + z^{12\beta} + z^{16\beta})$. Here each part has sum $\alpha + 41\beta$. Take $\nu = \alpha + 19\beta$, then A and $\overline{A}^\nu$ have no term in common.

1	300	2	396	8	392	117	389	18	382	105	398	6	295	10	394	11	388	14	384
120	381	119	285	113	289	4	292	103	299	16	283	115	386	111	287	110	293	107	297
21	280	22	376	28	372	137	369	38	362	125	378	26	275	30	374	31	368	34	364
160	341	159	245	153	249	44	252	143	259	56	243	155	346	151	247	150	253	147	257
201	100	202	196	208	192	317	189	218	182	305	198	206	95	210	194	211	188	214	184
320	181	319	85	313	89	204	92	303	99	216	83	315	186	311	87	310	93	307	97
221	80	222	176	228	172	337	169	238	162	325	178	226	75	230	174	231	168	234	164
340	161	339	65	333	69	224	72	323	79	236	63	335	166	331	67	330	73	327	77
261	40	262	136	268	132	377	129	278	122	365	138	266	35	270	134	271	128	274	124
360	141	359	45	353	49	244	52	343	59	256	43	355	146	351	47	350	53	347	57
41	260	42	356	48	352	157	349	58	342	145	358	46	255	50	354	51	348	54	344
140	361	139	265	133	269	24	272	123	279	36	263	135	366	131	267	130	273	127	277
61	240	62	336	68	332	177	329	78	322	165	338	66	235	70	334	71	328	74	324
180	321	179	225	173	229	64	232	163	239	76	223	175	326	171	227	170	233	167	237
81	220	82	316	88	312	197	309	98	302	185	318	86	215	90	314	91	308	94	304
200	301	199	205	193	209	84	212	183	219	96	203	195	306	191	207	190	213	187	217
241	60	242	156	248	152	357	149	258	142	345	158	246	55	250	154	251	148	254	144
380	121	379	25	373	29	264	32	363	39	276	23	375	126	371	27	370	33	367	37
281	20	282	116	288	112	397	109	298	102	385	118	286	15	290	114	291	108	294	104
400	101	399	5	393	9	284	12	383	19	296	3	395	106	391	7	390	13	387	17

Figure 8: 20x20 Franklin Magic Square $M_{20.1}$, constructed by method 1a

These partial solutions can be combined for $(\alpha, \beta, \gamma) = (100, 1, 20)$ or $(\alpha, \beta, \gamma) = (5, 20, 1)$. We obtain solution (x^1, y^1) with

$x^1 = (0, 100, 1, 4, 7, 8, 116, 11, 17, 18, 104, 2, 5, 105, 9, 6, 10, 12, 13, 16)$, and
$y^1 = (0, 280, 20, 240, 200, 80, 220, 60, 260, 40, 40, 260, 60, 220, 80, 200, 240, 20, 280, 0)$.

This yields matrix $M_{20.1}$, depicted in Figure 8. The second solution (x^2, y^2) is given by

$x^2 = (0, 5, 20, 80, 140, 160, 325, 220, 340, 360, 85, 40, 100, 105, 180, 120, 200, 240, 260, 320)$, and
$y^2 = (0, 14, 1, 12, 10, 4, 11, 3, 13, 2, 2, 13, 3, 11, 4, 10, 12, 1, 14, 0)$,

yielding matrix $M_{20.2}$. The last square is given in Figure 9.

6.2 Franklin Magic Squares of order $20 + 8k$

The construction of 20 by 20 squares given above can be extended to yield an n by n Franklin Magic Square for any $n = 20 + 8k$, with $k \geq 0$. Again we use method 1a.

1	395	21	320	141	240	326	180	341	40	86	360	101	295	181	280	201	160	261	80
386	20	366	95	246	175	61	235	46	375	301	55	286	120	206	135	186	255	126	335
2	394	22	319	142	239	327	179	342	39	87	359	102	294	182	279	202	159	262	79
388	18	368	93	248	173	63	233	48	373	303	53	288	118	208	133	188	253	128	333
11	385	31	310	151	230	336	170	351	30	96	350	111	285	191	270	211	150	271	70
396	10	376	85	256	165	71	225	56	365	311	45	296	110	216	125	196	245	136	325
12	384	32	309	152	229	337	169	352	29	97	349	112	284	192	269	212	149	272	69
397	9	377	84	257	164	72	224	57	364	312	44	297	109	217	124	197	244	137	324
14	382	34	307	154	227	339	167	354	27	99	347	114	282	194	267	214	147	274	67
398	8	378	83	258	163	73	223	58	363	313	43	298	108	218	123	198	243	138	323
3	393	23	318	143	238	328	178	343	38	88	358	103	293	183	278	203	158	263	78
387	19	367	94	247	174	62	234	47	374	302	54	287	119	207	134	187	254	127	334
4	392	24	317	144	237	329	177	344	37	89	357	104	292	184	277	204	157	264	77
389	17	369	92	249	172	64	232	49	372	304	52	289	117	209	132	189	252	129	332
5	391	25	316	145	236	330	176	345	36	90	356	105	291	185	276	205	156	265	76
390	16	370	91	250	171	65	231	50	371	305	51	290	116	210	131	190	251	130	331
13	383	33	308	153	228	338	168	353	28	98	348	113	283	193	268	213	148	273	68
399	7	379	82	259	162	74	222	59	362	314	42	299	107	219	122	199	242	139	322
15	381	35	306	155	226	340	166	355	26	100	346	115	281	195	266	215	146	275	66
400	6	380	81	260	161	75	221	60	361	315	41	300	106	220	121	200	241	140	321

Figure 9: 20x20 Franklin Magic Square $M_{20.2}$, constructed by method 1a

For B_0 we need a polynomial with $10+4k$ terms that can be split into two parts with equal exponent sum. We choose B_0 to be of the form $(1+z^{(10+4k)\gamma})(1+z^{\gamma}+\ldots+z^{(5+2k-1)\gamma})$, where the latter factor has $5+2k$ terms. Now a possible split into two parts may be $B_{00}=(1+z^{\gamma}+[z^{4\gamma}+z^{6\gamma}+\ldots+z^{(5+2k-3)\gamma}])+z^{(10+4k)\gamma}(1+[z^{\gamma}+z^{3\gamma}+\ldots+z^{(5+2k-2)\gamma}])$, and $B_{01}=(z^{2\gamma}+[z^{3\gamma}+z^{5\gamma}+\ldots+z^{(5+2k-2)\gamma}]+z^{(5+2k-1)\gamma})+z^{(10+4k)\gamma}(z^{2\gamma}+[z^{4\gamma}+z^{6\gamma}+\ldots+z^{(5+2k-1)\gamma}])$. Each part has exponent sum $(5+2k)(10+4k)\gamma/2+(5+2k)(5+2k-1)\gamma/2$.

For A to be derived from $A+\overline{A}^{n^2-1-\delta(B_0)}=(1+z^{(10+4k)\gamma})(1+z^{\alpha}+z^{2\alpha}+\ldots+z^{(n-1)\alpha})$, we can choose either $\gamma=1,\alpha=15+6k$, or $\alpha=1,\gamma=n=20+8k$. Now write $(1+z^{\alpha}+z^{2\alpha}+\ldots+z^{(n-1)\alpha})=(1+z^{\alpha}+z^{2\alpha}+\ldots+z^{(4k-1)\alpha})+z^{4k\alpha}(1+z^{\alpha}+z^{2\alpha}+\ldots+z^{(n-8k-1)\alpha})+z^{(n-4k)\alpha}(1+z^{\alpha}+z^{2\alpha}+\ldots+z^{(4k-1)\alpha})$.

Now define $A(z)=1\cdot(1+z^{\alpha}+z^{2\alpha}+\ldots+z^{(4k-1)\alpha})+z^{(10+4k)\gamma}\cdot z^{(n-4k)\alpha}(1+z^{\alpha}+z^{2\alpha}+\ldots+z^{(4k-1)\alpha})+z^{4k\alpha}A_{\gamma,\alpha}(z)$. Here the last part is taken from the general solution for $n=20$ in the previous subsection.

It is not difficult to see that both solutions generate an n by n Franklin Magic Square with the symmetry property along the horizontal middle line.

6.3 Huub Reijnders' method for a 20 by 20 Franklin Magic Square

The first known 20 by 20 Franklin square was constructed by Huub Reijnders, who did this apparently from scratch. It appears that his solution falls in the scheme set above. The exception is that he has a special way of solving $A+\overline{A}^{n^2-1-\delta(B_0)}=(1+z^{n/4})(1+z^{n}+z^{2n}+\ldots+z^{(n-1)n})$. His solution for $n=20$ is $A_{20}(z)=(1+z^5)(1+z^{20}+\ldots+z^{140})+(z^{160}+z^{180}+z^{200}+z^{220})$ which splits into $(1+z^{40}+z^{120}+z^{140})+z^{180}$, $z^5(1+z^{40}+z^{120}+z^{140})+z^{160}$, $(z^{20}+z^{60}+z^{80}+z^{100})+z^{220}$, and $z^5(z^{20}+z^{60}+z^{80}+z^{100})+z^{200}$, each with exponent sum 480.

The split for B_0 is the same as in the subsection above.

In terms of vectors Reijnders's solution is given by

$x=(0,2,1,3,10,4,11,12,13,14,14,13,12,11,4,10,3,1,2,0)$, and
$y=(0,5,60,65,80,85,220,200,120,125,140,145,180,160,100,105,40,45,20,25)$

yielding matrix $M_{20.r}$ given in Figure 10.

This solution approach can be extended to $n=20+8k$ by realizing that the above trick works by matching four exponents $n/4$ against one exponent n. In the remainder of solution A one needs only exponents that are multiples of n. This is easily realized by considering the solution $A_n(z)=(1+z^{n/4})(1+z^{n}+\ldots+z^{7n})+(z^{(n-4)n/2}+z^{(n-2)n/2}+z^{(n+0)n/2}+z^{(n+2)n/2})+Q_n(z)$ where $Q_n(z)=z^{8n}+z^{9n}+\ldots+z^{(n-6)n/2}+z^{(n+4)n/2}+\ldots+z^{(n-9)n}$. Note that $Q(z)$ contains $n-20=8k$ terms with an average exponent of $(n-1)n/2$. The terms in Q can be paired up in $4k$ pairs each with exponent sum $(n-1)n$, and these pairs can be evenly divided over four sets with equal exponent sum.

7 Almost-Franklin Magic Squares of order 12

It was proved in section 3 that no true Franklin Magic Squares of order 12 exist. Hence, one may try to construct Magic Squares that are as 'Franklin' as possible. We will stick to the

1	398	2	397	11	396	12	388	14	386	15	387	13	389	5	390	4	399	3	400
395	8	394	9	385	10	384	18	382	20	381	19	383	17	391	16	392	7	393	6
61	338	62	337	71	336	72	328	74	326	75	327	73	329	65	330	64	339	63	340
335	68	334	69	325	70	324	78	322	80	321	79	323	77	331	76	332	67	333	66
81	318	82	317	91	316	92	308	94	306	95	307	93	309	85	310	84	319	83	320
315	88	314	89	305	90	304	98	302	100	301	99	303	97	311	96	312	87	313	86
221	178	222	177	231	176	232	168	234	166	235	167	233	169	225	170	224	179	223	180
200	203	199	204	190	205	189	213	187	215	186	214	188	212	196	211	197	202	198	201
121	278	122	277	131	276	132	268	134	266	135	267	133	269	125	270	124	279	123	280
275	128	274	129	265	130	264	138	262	140	261	139	263	137	271	136	272	127	273	126
141	258	142	257	151	256	152	248	154	246	155	247	153	249	145	250	144	259	143	260
255	148	254	149	245	150	244	158	242	160	241	159	243	157	251	156	252	147	253	146
181	218	182	217	191	216	192	208	194	206	195	207	193	209	185	210	184	219	183	220
240	163	239	164	230	165	229	173	227	175	226	174	228	172	236	171	237	162	238	161
101	298	102	297	111	296	112	288	114	286	115	287	113	289	105	290	104	299	103	300
295	108	294	109	285	110	284	118	282	120	281	119	283	117	291	116	292	107	293	106
41	358	42	357	51	356	52	348	54	346	55	347	53	349	45	350	44	359	43	360
355	48	354	49	345	50	344	58	342	60	341	59	343	57	351	56	352	47	353	46
21	378	22	377	31	376	32	368	34	366	35	367	33	369	25	370	24	379	23	380
375	28	374	29	365	30	364	38	362	40	361	39	363	37	371	36	372	27	373	26

Figure 10: 20x20 Franklin Magic Square $M_{20.r}$, constructed by Reijnders

property of 2x2 squares having constant sum. Further we will stick to the typical Franklin feature of having bent diagonals with the magic sum. As order 12 Franklin Magic Squares do not exist we have to give up on having magic half rows and magic half columns. Actually we may stick to having Franklin half rows and Franklin bent-diagonals if we just give up Franklin half columns. Another opportunity is to have Franklin half rows and Franklin half columns and only horizontal Franklin bent-diagonals.

We may abandon the requirement of having magic half rows and magic half columns, and turn to having either the four-on-a-row property or having most-perfectness.

7.1 Horizontally correct Franklin Magic Squares

Application of method 1a yields a polynomial A of 24 terms and a polynomial B_0 of 6 terms. If $A + \overline{A}$ is of the form $(1 + z^{\alpha})(1 + z^{\beta} + \ldots + z^{11\beta})$, with $\alpha < \beta$ or $\alpha \geq 12\beta$, then a solution A exists that can be split into four parts of equal exponents sum. For instance $A(z) = (z^{3\beta} + z^{9\beta} + z^{5\beta+\alpha}) + (z^{\beta} + z^{5\beta} + z^{11\beta+\alpha}) + (z^{4\beta} + z^{10\beta} + z^{3\beta+\alpha}) + (z^{2\beta} + z^{11\beta} + z^{4\beta+\alpha})$. Here each part has exponent sum $\alpha + 17\beta$.

A matching B_0 of the form $(1 + z^{\delta})(1 + z^{\gamma} + z^{2\gamma})$ leads to a vector y, with $y_{11-i} = y_i$, and thus yields a square with magic bent-diagonals (both horizontally and vertically). Furthermore this square has a horizontal line of symmetry reflecting complementary entries. By properly ordering the exponents one even gets columns with the four-on-a-row property: take $y = (0, \delta, \delta+\gamma, \gamma, 2\gamma, \delta+2\gamma, \delta+2\gamma, 2\gamma, \gamma, \delta+\gamma, \delta, 0)$.

An example, with $\beta = 1$, $\alpha = 72$, $\gamma = 12$, $\delta = 36$, yields

$x = (3, 1, 9, 5, 77, 83, 4, 2, 10, 11, 75, 76)$ and
$y = (0, 36, 48, 12, 24, 60, 60, 24, 12, 48, 36, 0)$.

The result is the square M_{12a} given in Figure 11.

By interchanging rows $2, 4, 6$ with $8, 10, 12$ the square changes into one which has magic half columns, instead of having vertical magic bent-diagonals.

7.2 Decomposition and basic arrangements

In table 1 we list the possible decompositions of $\frac{z^{144}-1}{z-1}$ into two- and three term factors with coefficients 1. There are $\binom{6}{2} = 15$ of such decompositions.

They are labeled by a sequence of 2s and 3s that indicate the place of the factors with three terms.

Table 2 displays all possible permutations of numbers 0 up to 11 that have the properties

$$v_{4i} - v_{4i+1} + v_{4i+2} - v_{4i+3} = 0, \quad \text{for } i = 0, 1, 2, \tag{9}$$

$$v_0 + v_2 + v_4 = v_7 + v_9 + v_{11}, \tag{10}$$

up to isomorphism. These permutations were found by enumeration.

All rows except the ones marked by an asterisk have the property that for each pair $j, 11-j$ both entries are on an even position, or both are on an odd position.

Evidently, when properties (9) and (10) hold for a certain vector v, then they also hold for $w = Cv$, where C is an arbitrary scalar. The right-most entries $s - t$ in the table denote that for some vectors properties (9) and (10) also hold for exponents in the polynomials $(1 + z^{\alpha} + \ldots + z^{(s-1)\alpha})(1 + z^{\beta} + \ldots + z^{(t-1)\beta})$ according to the conversion table 3.

$$M_{12a} = \left|\begin{array}{cccccc|cccccc}
4 & 143 & 10 & 139 & 78 & 61 & 5 & 142 & 11 & 133 & 76 & 68 \\
105 & 38 & 99 & 42 & 31 & 120 & 104 & 39 & 98 & 48 & 33 & 113 \\
52 & 95 & 58 & 91 & 126 & 13 & 53 & 94 & 59 & 85 & 124 & 20 \\
129 & 14 & 123 & 18 & 55 & 96 & 128 & 15 & 122 & 24 & 57 & 89 \\
28 & 119 & 34 & 115 & 102 & 37 & 29 & 118 & 35 & 109 & 100 & 44 \\
81 & 62 & 75 & 66 & 7 & 144 & 80 & 63 & 74 & 72 & 9 & 137 \\
\hline
64 & 83 & 70 & 79 & 138 & 1 & 65 & 82 & 71 & 73 & 136 & 8 \\
117 & 26 & 111 & 30 & 43 & 108 & 116 & 27 & 110 & 36 & 45 & 101 \\
16 & 131 & 22 & 127 & 90 & 49 & 17 & 130 & 23 & 121 & 88 & 56 \\
93 & 50 & 87 & 54 & 19 & 132 & 92 & 51 & 86 & 60 & 21 & 125 \\
40 & 107 & 46 & 103 & 114 & 25 & 41 & 106 & 47 & 97 & 112 & 32 \\
141 & 2 & 135 & 6 & 67 & 84 & 140 & 3 & 134 & 12 & 69 & 77
\end{array}\right|$$

Figure 11: As Franklin as possible, no magic half columns

222233	$(1+z)(1+z^2)(1+z^4)(1+z^8)(1+z^{16}+z^{32})(1+z^{48}+z^{96})$
222323	$(1+z)(1+z^2)(1+z^4)(1+z^8+z^{16})(1+z^{24})(1+z^{48}+z^{96})$
222332	$(1+z)(1+z^2)(1+z^4)(1+z^8+z^{16})(1+z^{24}+z^{48})(1+z^{72})$
223223	$(1+z)(1+z^2)(1+z^4+z^8)(1+z^{12})(1+z^{24})(1+z^{48}+z^{96})$
223232	$(1+z)(1+z^2)(1+z^4+z^8)(1+z^{12})(1+z^{24}+z^{48})(1+z^{72})$
223322	$(1+z)(1+z^2)(1+z^4+z^8)(1+z^{12}+z^{24})(1+z^{36})(1+z^{72})$
232223	$(1+z)(1+z^2+z^4)(1+z^6)(1+z^{12})(1+z^{24})(1+z^{48}+z^{96})$
232232	$(1+z)(1+z^2+z^4)(1+z^6)(1+z^{12})(1+z^{24}+z^{48})(1+z^{72})$
232322	$(1+z)(1+z^2+z^4)(1+z^6)(1+z^{12}+z^{24})(1+z^{36})(1+z^{72})$
233222	$(1+z)(1+z^2+z^4)(1+z^6+z^{12})(1+z^{24})(1+z^{48})(1+z^{96})$
322223	$(1+z+z^2)(1+z^3)(1+z^6)(1+z^{12})(1+z^{24})(1+z^{48}+z^{96})$
322232	$(1+z+z^2)(1+z^3)(1+z^6)(1+z^{12})(1+z^{24}+z^{48})(1+z^{72})$
322322	$(1+z+z^2)(1+z^3)(1+z^6)(1+z^{12}+z^{24})(1+z^{36})(1+z^{72})$
323222	$(1+z+z^2)(1+z^3)(1+z^6+z^{12})(1+z^{18})(1+z^{36})(1+z^{72})$
332222	$(1+z+z^2)(1+z^3+z^6)(1+z^9)(1+z^{18})(1+z^{36})(1+z^{72})$

Table 1: Possible decompositions of $\frac{z^{144}-1}{z-1}$

v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	remark
0	1	7	6	11	8	2	5	4	3	9	10	4-3, 6-2
0	1	7	6	9	10	4	3	2	5	11	8	6-2
0	1	10	9	6	7	4	3	2	5	11	8	*
0	2	6	4	11	10	8	9	3	1	5	7	3-4
0	2	8	6	7	10	4	1	3	5	11	9	4-3
0	2	8	6	10	7	1	4	3	5	11	9	4-3
0	2	10	8	4	9	11	6	1	3	7	5	
0	2	10	8	6	7	5	4	1	3	11	9	3-4
0	2	10	8	7	6	4	5	1	3	11	9	3-4
0	3	9	6	11	4	1	8	2	5	10	7	2-6
0	3	9	6	10	5	2	7	1	4	11	8	2-6
0	3	9	6	7	8	2	1	4	5	11	10	6-2
0	3	9	6	7	8	5	4	1	2	11	10	*
0	3	9	6	7	8	11	10	2	1	4	5	
0	3	10	7	9	4	1	6	2	5	11	8	2-6
2	5	11	8	0	7	10	3	1	4	9	6	2-6
4	5	11	10	0	3	9	6	2	1	7	8	4-3, 6-2
4	6	10	8	0	5	7	2	1	3	11	9	
5	7	11	9	0	1	3	2	6	4	8	10	3-4
6	7	10	9	1	0	3	4	5	2	8	11	

Table 2: Possible arrangements with 4-on-a-row and bent-diagonal properties

s	t	conversion of k
2	6	$\lfloor k/6 \rfloor \alpha + (k\%6)\beta$
3	4	$\lfloor k/4 \rfloor \alpha + (k\%4)\beta$
4	3	$\lfloor k/3 \rfloor \alpha + (k\%3)\beta$
6	2	$\lfloor k/2 \rfloor \alpha + (k\%2)\beta$

Table 3: Conversion table for order 12 sequences

$M_{12.V} =$

56	17	129	88	19	90	127	54	92	125	21	52
86	131	13	60	123	58	15	94	50	23	121	96
68	5	141	76	31	78	139	42	104	113	33	40
80	137	7	66	117	64	9	100	44	29	115	102
41	32	114	103	4	105	112	69	77	140	6	67
107	110	34	39	144	37	36	73	71	2	142	75
47	26	120	97	10	99	118	63	83	134	12	61
95	122	22	51	132	49	24	85	59	14	130	87
53	20	126	91	16	93	124	57	89	128	18	55
101	116	28	45	138	43	30	79	65	8	136	81
62	11	135	82	25	84	133	48	98	119	27	46
74	143	1	72	111	70	3	106	38	35	109	108

Figure 12: 12x12 Magic Square with bent-diagonals and 4-on-a-row, by method 2

7.3 Method 2 for bent-diagonal and 4-on-a-row properties

Application of method 2 on any vector x taken from Table 2, together with a vector y obtained by taking any row of this table and multiplying it by 12 directly leads to a pan-diagonal 12x12 Magic Square with the bent-diagonals property as well as the four-on-a-row property. One should not take any of the rows marked by an asterisk.

Now we show how method 2 can be applied on a less trivial factorization. Consider the decomposition $\frac{z^{144}-1}{z-1} = A(z)B(z)$, with $A(z) = (1+z+z^2)(1+z^{36}+z^{72}+z^{108})$ and $B(z) = (1+z^3+z^6+\ldots+z^{33})$. For B any row from the table not marked by an asterisk, multiplied by 3 will do. Let us take the last one: $y = (18, 21, 30, 27, 3, 0, 9, 12, 15, 6, 24, 33)$. For A pick a row marked 3-4 or 4-3, let us say the one but last row. We have $\alpha = 1$, $\beta = 36$, $s = 3$, $t = 4$. The row is converted to $x = (1+36, 1+108, 2+108, 2+36, 0+0, 0+36, 0+108, 0+72, 1+72, 1+0, 2+0, 2+72) = (37, 109, 110, 38, 0, 36, 108, 72, 73, 1, 2, 74)$. The resulting square $M_{12.V}$ is depicted in Figure 12.

Similarly, an even more complicated decomposition can be base of a pan-diagonal 12x12 square with 4-on-a-row and bent-diagonal properties. Consider any decomposition of $\frac{z^{144}-1}{z-1}$ into four factors, each with a geometric series of 2, 3, 4 or 6 terms. For example, take $A(z) = (1+z)(1+z^6+z^{12}+z^{18}+z^{24}+z^{30})$ and $B(z) = (1+z^2+z^4)(1+z^{36}+z^{72}+z^{108})$. For an appropriate vector x select a row from table 2 marked 2-6 or 6-2, for a vector y take a row with mark 3-4 or 4-3. Using the conversion table 3 one constructs x and y and from these one builds a 12x12 square with the desired properties.

v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	remark
0	2	8	6	7	10	10	7	6	8	2	0	4-3
0	2	10	8	6	7	7	6	8	10	2	0	3-4

Table 4: Arrangements of B with 4-on-a-row, bent-diagonal and symmetry properties

7.4 Method 1b for bent-diagonal and 4-on-a-row properties

Using method 1b we start again from a decomposition into four factors as above. Given the decomposition select two factors with $3+4$ or $2+6$ terms, whose product will be A, and use the conversion table 3 to build an appropriate vector x.

The other two factors will have as product $B_0+\overline{B_0}^{143-\delta(A)}$. Choose B_0 in such a way that the six terms have exponents two pairs of which have the same sum. This is often possible in many ways. Let $e_0,\dots,e_5$ be the exponents in B_0 and assume $e_0+e_1=e_2+e_3$. Define $y=(e_0,e_2,e_1,e_3,e_4,e_5,e_5,e_4,e_3,e_1,e_2,e_0)$. This arrangement will yield a square which has bent-diagonal and 4-on-a-row properties, and in addition, it will have the mirroring property, i.e. complementary entries will reflect in the horizontal axis of symmetry.

In general this procedure will not yield a square which is pan-diagonal. If we want to enforce this property we have to be more restrictive in the choice for A and $B_0+\overline{B_0}^{143-\delta(A)}$. In particular we need that the average exponent in B_0 equals half the degree of $B_0+\overline{B_0}^{143-\delta(A)}$.

The only basic six-term that has the desired property for B_0 (with $\delta(A)=132$) is $z^0+z^2+z^6+z^7+z^8+z^{10}$. Note that there are two ways of pairing these exponents up appropriately. Either take $e^1=(0,8,2,6,7,10)$ or $e^2=(0,10,2,8,6,7)$. Now the basic y-vectors, with their potential conversions are given in table 4.

A general description to generate a pan-diagonal Magic Square of order 12, with bent-diagonal property, with four-on-a-row property and which reflects along the horizontal axis of symmetry is the following:

1. From decomposition table 1 select a row, and pick two consecutive two-term factors. Multiply them to get a factor $(1+z^\alpha+z^{2\alpha}+z^{3\alpha})$;

2. From the same row select a three-term factor $(1+z^\beta+z^{2\beta})$ such that of the three remaining factors at least two are consecutive;

3. These three remaining factors constitute a polynomial $A(z)$ for which there are several possible arrangements, by use of table 2 and an appropriate conversion. Rows marked with an asterisk should not be considered;

4. The other factors make up the factor $B_0+\overline{B_0}(z)$ to be arranged as one of the rows in table 4.

Remark: the HSA-square, designed by a group of Dutch high school students and publicized in March 2007, fits in this scheme. As an example, let us select the second row in the decomposition table $B_0+\overline{B_0}^{\nu}=(1+z^2)(1+z^4)(1+z^{48}+z^{96})$ and $A(z)=(1+z)(1+z^8+z^{16})(1+z^{24})$. Writing out the consecutive factors we obtain $A(z)=(1+z)(1+z^8+z^{16}+z^{24}+z^{32}+z^{40})$ and $B_0+\overline{B_0}^{\nu}=(1+z^2+z^4+z^6)(1+z^{48}+z^{96})$, with $\nu=143-41=102$. From table 2 pick the first row: $0,1,7,6,11,8,2,5,4,3,9,10$ to arrange the exponents of

$$
M_{pd12.4} = \left|\begin{array}{rrrr|rrrr|rrrr}
\hline
1 & 143 & 26 & 120 & 42 & 112 & 9 & 127 & 17 & 135 & 34 & 104 \\
48 & 98 & 23 & 121 & 7 & 129 & 40 & 114 & 32 & 106 & 15 & 137 \\
101 & 43 & 126 & 20 & 142 & 12 & 109 & 27 & 117 & 35 & 134 & 4 \\
140 & 6 & 115 & 29 & 99 & 37 & 132 & 22 & 124 & 14 & 107 & 45 \\
\hline
53 & 91 & 78 & 68 & 94 & 60 & 61 & 75 & 69 & 83 & 86 & 52 \\
90 & 56 & 65 & 79 & 49 & 87 & 82 & 72 & 74 & 64 & 57 & 95 \\
55 & 89 & 80 & 66 & 96 & 58 & 63 & 73 & 71 & 81 & 88 & 50 \\
92 & 54 & 67 & 77 & 51 & 85 & 84 & 70 & 76 & 62 & 59 & 93 \\
\hline
5 & 139 & 30 & 116 & 46 & 108 & 13 & 123 & 21 & 131 & 38 & 100 \\
44 & 102 & 19 & 125 & 3 & 133 & 36 & 118 & 28 & 110 & 11 & 141 \\
97 & 47 & 122 & 24 & 138 & 16 & 105 & 31 & 113 & 39 & 130 & 8 \\
144 & 2 & 119 & 25 & 103 & 33 & 136 & 18 & 128 & 10 & 111 & 41 \\
\hline
\end{array}\right|
$$

Figure 13: Pan-diagonal symmetric 12x12 Magic Square with bent-diagonals, by method 1b

A. It has a 6-2 generalization, with $\alpha = 8$ and $\beta = 1$. We obtain $x = (0\alpha + 0\beta, 0\alpha + 1\beta, 3\alpha + 1\beta, 3\alpha + 0\beta, 5\alpha + 1\beta, 4\alpha + 0\beta, 1\alpha + 0\beta, 2\alpha + 1\beta, 2\alpha + 0\beta, 1\alpha + 1\beta, 4\alpha + 1\beta, 5\alpha + 0\beta)$ $= (0, 1, 25, 24, 41, 32, 8, 17, 16, 9, 33, 40)$.

To build B_0 pick the first row from table 4 $0, 2, 8, 6, 7, 10, 10, 7, 6, 8, 2, 0$. With the 4-3 factorization with $\alpha = 2$ and $\beta = 48$ this leads to $y = (0\alpha+0\beta, 0\alpha+2\beta, 2\alpha+2\beta, 2\alpha+0\beta, 2\alpha+1\beta, 3\alpha+1\beta, 3\alpha+1\beta, 2\alpha+1\beta, 2\alpha+0\beta, 2\alpha+2\beta, 0\alpha+2\beta, 0\alpha+0\beta) = (0, 96, 100, 4, 52, 54, 54, 52, 4, 100, 96, 0)$.

Plugging in these vectors yields a square $M_{pd12.4}$ depicted in Figure 13.

7.5 Method 1a for bent-diagonal and 4-on-a-row properties

Using method 1a we start from a decomposition into factors $A+\overline{A}$ and B_0, where the first has 24 terms and the second only 6. If we take for the first factor $1+z^\alpha$ times a factor representable (by conversion) with a row from table 2, we can take for A this second factor. If $B_0(z) = (1+z^\beta)(1+z^\gamma+z^{2\gamma})$, a proper reordering gives $B_0(z) = (1+z^{\beta+\gamma})+(z^{2\gamma}+z^{2\gamma+\beta})+(z^\gamma+z^\beta)$ and $B_1(z) = (z^\beta + z^\gamma) + (z^{2\gamma+\beta} + z^{2\gamma}) + (z^{\beta+\gamma} + 1)$. The resulting square will have bent-diagonal properties, four-on-a-row properties and symmetry along the horizontal axis of symmetry. The result will in general not be pan-diagonal.

To enforce pan-diagonality, the choice for $A + \overline{A}^{143-\delta(B_0)}$ is restricted to be of the form $(1 + z^\alpha)$ times a 12-term representable by a row from table 4.

7.6 Method 2 for constructing most-perfect order 12 Magic Squares

It is possible to impose on the 12 by 12 Magic Square that it has the most-perfectness property. For this to be true one has to have $x_j + x_{j+6}$ equal to $\delta(A)$. Such an arrangement for $A(z) = 1 + z + \cdots + z^{11}$ can explicitly be found by complete enumeration.

v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	remark
0	2	6	7	8	10	11	9	5	4	3	1	3-4, 4-3
0	2	6	7	10	8	11	9	5	4	1	3	3-4, 4-3
0	2	6	8	7	10	11	9	5	3	4	1	3-4, 4-3
0	2	7	6	8	10	11	9	4	5	3	1	3-4, 4-3
0	2	7	6	10	8	11	9	4	5	1	3	3-4, 4-3
0	2	8	6	10	7	11	9	3	5	1	4	3-4, 4-3
0	6	2	7	8	10	11	5	9	4	3	1	3-4, 4-3
0	6	2	7	10	8	11	5	9	4	1	3	3-4, 4-3
0	6	2	8	7	10	11	5	9	3	4	1	3-4, 4-3
0	7	2	8	6	10	11	4	9	3	5	1	3-4, 4-3

Table 5: arrangement with most-perfect features

This yields the table 5, in which the remark section, as before, indicates how to use the conversion table 3 to get even more polynomials with the property of providing a most-perfect arrangement.

As an example, take the first row and 12 times the last row of Table 5 to get

$x = (0, 2, 6, 7, 8, 10, 11, 9, 5, 4, 3, 1)$, and
$y = (0, 84, 24, 96, 72, 120, 132, 48, 108, 36, 60, 12)$.

The resulting square $M_{12.p}$ has magic row and column sums, magic bent-diagonals, and has complementary entries in opposite quadrants, as seen from Figure 14.

8 Conclusions

The existence of Franklin Magic Squares of order $n = 4k$, with $n \neq 4$ and $n \neq 12$ has been shown. Multiples of 8 pose no problems. Orders $20 + 8k$ are more difficult to realize, but not impossible. We have described four methods by which one can construct many Franklin Magic Squares. We are not aware of any Franklin Magic Square that does not fit into one of these four schemes.

The non-existence of a 12 by 12 Franklin Magic Square has been demonstrated by an exhaustive search that was only possible by maximal use of symmetry arguments as well as aggressive pruning.

I like to thank Andries Brouwer and Tonny Hurkens for fruitful discussions, and of course Arno van den Essen, and students Petra, Jesse and Willem, for the hype and interest they created.

$M_{12.p} =$

1	142	7	137	9	134	12	135	6	140	4	143
60	87	54	92	52	95	49	94	55	89	57	86
25	118	31	113	33	110	36	111	30	116	28	119
48	99	42	104	40	107	37	106	43	101	45	98
73	70	79	65	81	62	84	63	78	68	76	71
24	123	18	128	16	131	13	130	19	125	21	122
133	10	139	5	141	2	144	3	138	8	136	11
96	51	90	56	88	59	85	58	91	53	93	50
109	34	115	29	117	26	120	27	114	32	112	35
108	39	102	44	100	47	97	46	103	41	105	38
61	82	67	77	69	74	72	75	66	80	64	83
132	15	126	20	124	23	121	22	127	17	129	14

Figure 14: Most-perfect 12 by 12 Magic Square with bent-diagonals